

QED effective action in Krein space quantization

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Abstract

The one-loop effective action of QED is calculated by the Schwinger method in Krein space quantization. We show that the effective action is naturally finite and regularized. It also coincides with the renormalized solution which was derived by Schwinger.

Keywords: Krein space, Effective action

1 Introduction

The minimally coupled scalar field in de Sitter space plays an important role in the inflationary model as well as in the linear quantum gravity. In 1985, Allen showed that a covariant quantization of minimally coupled scalar field cannot be constructed from positive norm states alone [1]. For obtaining a covariant quantization of this field, a new method of field quantization has been presented, *i.e.* Krein space quantization [2]. It has been proven that the use of the two sets of solutions (positive and negative norm states) are an unavoidable feature for preservation of (1) causality (locality), (2) covariance, and (3) elimination of the infrared divergence for the minimally coupled scalar field in de Sitter space. The most interesting result of this construction is the convergence of the Green function at large distances, which means that the infrared divergence is gauge dependent [3].

It has been shown that quantization in Krein space removes all ultraviolet divergences of QFT except the light cone singularity [2, 3, 4]. It was conjectured that quantum metric fluctuations might smear out the singularities of Green functions on the light cone, but it does not remove other ultraviolet divergences [5]. However, by using the Krein space quantization and the quantum metric fluctuations in the linear approximation, we showed that the problem of infinities in QFT disappears [6].

This method was applied to different problem and the natural regularized results were obtained [7, 8]. We have computed the scalar field effective action in Krein space quantization which includes quantum metric fluctuation [9]. A finite result is obtained naturally with the

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same physical results as yielded by the standard method. We hope this method can solve the problem of non-renormalizability of quantum gravity in the linear approximation.

In this paper, we address the problem of derivation of the low-energy effective action, which is solved perturbatively for QED [10]. Using the Krein space method and quantum metric fluctuation at the linear approximation, we calculate the one loop effective action for QED. We have shown that the result not only is regularized but also is equivalent to the renormalized result which was reported by Schwinger.

The paper is organized as follows. In section 2, we briefly recall the propagator derivation for scalar field in Krein space quantization. Section 3 is devoted to express the one-loop effective action in terms of the new propagator in our method. Then, we develop the Schwinger technique for generating the perturbative expression in section 4. In this section we present the main results of our paper. In appendices, we have provided some calculation techniques which are needed.

2 Scalar Green function

We review the elementary facts about Krein space quantization. A classical scalar field $\phi(x)$ satisfies the following field equation

$$(\square + m^2)\phi(x) = 0 = (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(x), \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (2.1)$$

Inner (*Klein-Gordon*) product and related norms are defined by [11]

$$(\phi_1, \phi_2) = -i \int_{t=\text{const.}} \phi_1(x) \overleftrightarrow{\partial}_t \phi_2^*(x) d^3x. \quad (2.2)$$

Two sets of solutions are given by:

$$u_p(k, x) = \frac{e^{i\vec{k}\cdot\vec{x}-iwt}}{\sqrt{(2\pi)^3 2w}} = \frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2w}}, \quad (2.3)$$

$$u_n(k, x) = \frac{e^{-i\vec{k}\cdot\vec{x}+iwt}}{\sqrt{(2\pi)^3 2w}} = \frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3 2w}}, \quad (2.4)$$

where $w(\vec{k}) = k^0 = (\vec{k}\cdot\vec{k} + m^2)^{\frac{1}{2}} \geq 0$, note that u_n has the negative norm. In Krein space the quantum field is defined as follows [4]:

$$\phi(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)], \quad (2.5)$$

where

$$\phi_p(x) = \int d^3\vec{k} [a(\vec{k})u_p(k, x) + a^\dagger(\vec{k})u_p^*(k, x)],$$

$$\phi_n(x) = \int d^3\vec{k} [b(\vec{k})u_n(k, x) + b^\dagger(\vec{k})u_n^*(k, x)].$$

$a(\vec{k})$ and $b(\vec{k})$ are two independent operators. The time-ordered product propagator for this field operator is

$$iG_T(x, x') = \langle 0 | T\phi(x)\phi(x') | 0 \rangle = \theta(t - t')\mathcal{W}(x, x') + \theta(t' - t)\mathcal{W}(x', x). \quad (2.6)$$

In this case we obtain

$$G_T(x, x') = \frac{1}{2}[G_F(x, x') + (G_F(x, x'))^*] = \Re G_F(x, x'), \quad (2.7)$$

where the Feynman Green function is defined by [11]

$$\begin{aligned} G_F(x, x') &= \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - x')} \tilde{G}_F(p) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x - x')}}{p^2 - m^2 + i\epsilon} \\ &= -\frac{1}{8\pi} \delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0}) - iN_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} \\ &\quad - \frac{im^2}{4\pi^2} \theta(-\sigma_0) \frac{K_1(\sqrt{-2m^2\sigma_0})}{\sqrt{-2m^2\sigma_0}}, \end{aligned} \quad (2.8)$$

where $\sigma_0 = \frac{1}{2}(x - x')^2$. So we have

$$\begin{aligned} G_T(x, x') &= \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - x')} \mathcal{P}\mathcal{P} \frac{1}{p^2 - m^2} \\ &= -\frac{1}{8\pi} \delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad x \neq x', \end{aligned} \quad (2.9)$$

$\mathcal{P}\mathcal{P}$ stands for the principal parts. Contribution of the coincident point singularity ($x = x'$) merely appears in the imaginary part of G_F ([3] and equation (9.52) in [11])

$$G_F(x, x) = -\frac{2i}{(4\pi)^2} \frac{m^2}{d - 4} + G_F^{\text{finit}}(x, x),$$

where d is the space-time dimension and $G_F^{\text{finit}}(x, x)$ becomes finite as $d \rightarrow 4$. Note that the singularity of the Eq.(2.9) takes place only on the cone *i.e.*, $x \neq x', \sigma_0 = 0$.

It has been shown that the quantum metric fluctuations remove the singularities of Green's functions on the light cone [5]. Therefore, the quantum field theory in Krein space, including the quantum metric fluctuation ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), removes all the ultraviolet divergencies of the theory [6, 5], so one can write:

$$\langle G_T(x, x') \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} \exp\left(-\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}\right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad (2.10)$$

where $2\sigma = g_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)$ and σ_1 is the first order shift in σ , due to the linear quantum gravity ($\sigma = \sigma_0 + \sigma_1 + O(h^2)$). The average value is taken over the quantum metric fluctuation and in the case of $2\sigma_0 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) = 0$ we have $\langle\sigma_1^2\rangle \neq 0$. So, we get

$$\langle G_T(0) \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} + \frac{m^2}{8\pi} \frac{1}{2}. \quad (2.11)$$

It should be noted that $\langle \sigma_1^2 \rangle$ is related to the density of gravitons [5].

By using the Fourier transformation of Dirac delta function,

$$-\frac{1}{8\pi}\delta(\sigma_0) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \mathcal{P}\mathcal{P} \frac{1}{p^2},$$

or equivalently

$$\frac{1}{8\pi^2} \frac{1}{\sigma_0} = - \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \pi \delta(p^2),$$

for the second part of Green function, we obtain

$$\frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.12)$$

And for the first part we have

$$-\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} \exp \left[-\frac{(x-x')^4}{4\langle \sigma_1^2 \rangle} \right] = \int \frac{d^4p}{(2\pi)^4} e^{-ik \cdot (x-x')} \tilde{G}_1(p),$$

where \tilde{G}_1 is fourier transformation of the first part of the Green function (2.10). Therefore, we obtain

$$\langle \tilde{G}_T(p) \rangle = \tilde{G}_1(p) + \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.13)$$

In the previous paper, we proved that in the one-loop approximation, the Green function in Krein space quantization, which appears in the transition amplitude is [4]:

$$\langle \tilde{G}_T(p) \rangle |_{\text{one-loop}} \equiv \tilde{G}_T(p) |_{\text{one-loop}} \equiv \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.14)$$

That means in the one loop approximation, the contribution of $\tilde{G}_1(p)$ is negligible. It is worth to mention that in order to improve the UV behavior in relativistic higher-derivative correction theories, the propagator (2.14) has been used by some authors [12, 13]. It is also appear in supersymmetry (equation (20.76) in [14]).

3 One-loop effective action

Let us start from the general method which was originally developed in [9]. The one-loop effective action in QED reduces to computing the fermion determinant

$$\begin{aligned} J &= \frac{i}{2} Tr \ln \left[1 - \frac{2eA \cdot p + \frac{\epsilon}{2} \sigma_{\mu\nu} F^{\mu\nu} - e^2 A^2}{p^2 - m^2 + i\epsilon} \right] \\ &= \frac{i}{2} \int d^4x \langle x | \ln \left[1 - \frac{2eA \cdot p + \frac{\epsilon}{2} \sigma_{\mu\nu} F^{\mu\nu} - e^2 A^2}{p^2 - m^2 + i\epsilon} \right] | x \rangle. \end{aligned} \quad (3.1)$$

One can write this determinant, in proper time method, as [15]

$$\begin{aligned}
J &= \frac{i}{2} \int_0^\infty ds s^{-1} e^{-ism^2} \text{Tr} \exp \left\{ -is[p^2 - e(p.A + A.p) - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} + e^2 A^2] \right\} \\
&- \frac{i}{2} \int_0^\infty ds s^{-1} e^{-ism^2} \text{Tr} \exp(-isp^2) \\
&= \frac{i}{2} \int_0^\infty ds s^{-1} e^{-ism^2} [\text{Tr} U(s) - \text{Tr} U_0(s)],
\end{aligned} \tag{3.2}$$

where $U(s) = \exp \left\{ -is[p^2 - e(p.A + A.p) - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} + e^2 A^2] \right\}$ and $U_0(s) = \exp(-isp^2)$. In Krein space quantization including the quantum metric fluctuation, equation (3.1) reads as

$$J_{kr} = \frac{i}{2} \text{Tr} \ln \left[1 - (2eA.p + \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} - e^2 A^2) \mathcal{P} \mathcal{P} \frac{m^2}{p^2(p^2 - m^2)} \right]. \tag{3.3}$$

If we take

$$V = \frac{1}{2} m^2 (2eA.p + \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} - e^2 A^2),$$

we can write

$$\begin{aligned}
J_{kr} &= \frac{i}{2} \text{Tr} \ln \left[1 - \mathcal{P} \mathcal{P} \frac{2V}{p^2(p^2 - m^2)} \right] \\
&= \frac{i}{2} \text{Tr} \ln \left[1 - V \left(\frac{1}{p^2(p^2 - m^2) + i\epsilon} + \frac{1}{p^2(p^2 - m^2) - i\epsilon} \right) \right] \\
&= \frac{i}{2} \text{Tr} \ln \left[\left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right) - \left(\frac{V}{p^2(p^2 - m^2)} \right)^2 \right],
\end{aligned} \tag{3.4}$$

where ϵ^2 has been vanished. By continuing the calculation, for this equation we obtain

$$\begin{aligned}
J_{kr} &= \frac{i}{2} \text{Tr} \ln \left[\left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right) \right. \\
&\quad \times \left. \left(1 - \frac{\left(\frac{V}{p^2(p^2 - m^2)} \right)^2}{\left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right)} \right) \right] \\
&= \frac{i}{2} \text{Tr} \ln \left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) + \frac{i}{2} \text{Tr} \ln \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right) \\
&\quad + \frac{i}{2} \text{Tr} \ln \left[1 - \left(\frac{\left(\frac{V}{p^2(p^2 - m^2)} \right)^2}{\left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right)} \right) \right].
\end{aligned} \tag{3.5}$$

The last term in this equation splits into two terms. So, J_{kr} becomes

$$\begin{aligned}
J_{kr} &= \frac{i}{2} \text{Tr} \ln \left(1 - \frac{V}{p^2(p^2 - m^2) - i\epsilon} \right) + \frac{i}{2} \text{Tr} \ln \left(1 - \frac{V}{p^2(p^2 - m^2) + i\epsilon} \right) \\
&\quad + \frac{i}{2} \text{Tr} \ln \left(1 - \frac{V}{p^2(p^2 - m^2) - V} \right) + \frac{i}{2} \text{Tr} \ln \left(1 + \frac{V}{p^2(p^2 - m^2) - V} \right).
\end{aligned} \tag{3.6}$$

By using the results in appendix B, we obtain:

$$J_{kr} = \frac{i}{2}Tr \ln \left(1 + \frac{Y}{p^2} \right) + \frac{i}{2}Tr \ln \left(1 - \frac{Y}{p^2 - m^2} \right) + \frac{i}{2}Tr \ln \left(1 + \frac{Y^2}{(m^2 p^2 + Y)(m^2 p^2 - m^4 - Y)} \right), \quad (3.7)$$

where $V = \frac{m^2 Y}{2}$ and $Y = 2eA.P + \frac{\epsilon}{2}\sigma_{\mu\nu}F^{\mu\nu} - e^2 A^2$. We define the functions J , J_0 and J_1 as follow:

$$J = \frac{i}{2}Tr \ln \left(1 - \frac{Y}{p^2 - m^2} \right), \quad J_0 = \frac{i}{2}Tr \ln \left(1 + \frac{Y}{p^2} \right),$$

$$J_1 = \frac{i}{2}Tr \ln \left(1 + \frac{Y^2}{(m^2 p^2 + Y)(m^2 p^2 - m^4 - Y)} \right),$$

so, we have

$$J_{kr} = J + J_0 + J_1. \quad (3.8)$$

4 Regularized effective action

The approximate evaluation of J_{kr} is the main purpose of this article. Therefore, following the perturbation approach developed in [10], we discuss the evaluations of J , J_0 and J_1 in equation (3.8), which lead to derive the former expression for the one loop effective action in an external field. It's no difficult to show that

$$\begin{aligned} J &= \frac{i}{2}Tr \ln \left(1 - \frac{2eA.P + \frac{\epsilon}{2}\sigma_{\mu\nu}F^{\mu\nu} - e^2 A^2}{p^2 - m^2 + i\epsilon} \right) \\ &= \frac{i}{2} \int_0^\infty ds s^{-1} e^{-ism^2} [Tr U(s) - Tr U_0(s)] \\ &= W^{(1)} - \frac{i}{2} \int_0^\infty ds s^{-1} e^{-ism^2} Tr U_0(s) \\ &= -\frac{e^2}{12\pi^2} \int_0^\infty ds s^{-1} \exp(-m^2 s) \int d^4 k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ &\quad + \frac{e^2}{4\pi^2} \int d^4 k \frac{k^2}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{m^2 + \frac{k^2}{4}(1 - v^2)}, \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} J_0 &= \frac{i}{2}Tr \ln \left(1 + \frac{2eA.P + \frac{\epsilon}{2}\sigma_{\mu\nu}F^{\mu\nu} - e^2 A^2}{p^2 + i\epsilon} \right) \\ &= -\frac{e^2}{4\pi^2} \int_0^\infty ds s^{-2} \int d^4 k A_\mu(-k) A_\mu(k) \\ &\quad - \frac{e^2}{12\pi^2} \int_0^\infty ds s^{-1} \int d^4 k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ &\quad + \frac{e^2}{16\pi^2} \int d^4 k F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 v^2 dv \frac{1 - \frac{1}{3}v^2}{1 - v^2}. \end{aligned} \quad (4.2)$$

We have employed the methods which are presented in appendix A. In J_1 , we expand the logarithm function and keep the first term

$$\begin{aligned} J_1 &= \frac{i}{2} \text{Tr} \ln \left(1 + \frac{Y^2}{(p^2 + Y)(p^2 - m^2 - Y)} \right) \simeq \frac{i}{2} \text{Tr} \frac{Y^2}{(p^2 + Y)(p^2 - m^2 - Y)} \\ &= -\frac{i}{2} \text{Tr} \frac{Y^2}{m^2 + 2Y} \left(\frac{1}{p^2 + Y} - \frac{1}{p^2 - m^2 - Y} \right), \end{aligned} \quad (4.3)$$

and so we can write

$$\begin{aligned} J_1 &= -\frac{i}{2} \text{Tr} \frac{Y^2}{i(m^2 + 2Y)} \int_0^\infty ds e^{-isp^2} (e^{-isY} - e^{is(Y+m^2)}), \\ &= -\frac{i}{2} \text{Tr} \frac{Y^2}{im^2} \left[1 + \sum_{n=1} \left(-\frac{2Y}{m^2} \right)^n \right] \int_0^\infty ds e^{-isp^2} (e^{-isY} - e^{is(Y+m^2)}). \end{aligned} \quad (4.4)$$

We restrict ourselves to a specific finite number of the powers of Y to compare with the expressions of J and J_0 , hence, we have

$$J_1 \simeq \frac{i}{2} \text{Tr} \int_0^\infty ds s Y^2 e^{-isp^2}. \quad (4.5)$$

One can write [Appendix A]:

$$\begin{aligned} J_1 &\simeq \frac{ie^2}{2} \int_0^\infty ds s \\ &\times \left(\frac{1}{2} \int_{-1}^1 dv \text{Tr} [(p.A + A.p) \exp(-ip^2 \frac{1}{2}(1-v)s) \times (p.A + A.p) \exp(-ip^2 \frac{1}{2}(1+v)s)] \right. \\ &\left. + \frac{1}{2} \int_{-1}^1 dv \text{Tr} [\frac{1}{2} \sigma F \exp(-ip^2 \frac{1}{2}(1-v)s) \times \frac{1}{2} \sigma F \exp(-ip^2 \frac{1}{2}(1+v)s)] \right). \end{aligned} \quad (4.6)$$

Now, we calculate these traces in a momentum representation

$$\begin{aligned} J_1 &\simeq \frac{2ie^2}{(2\pi)^4} \int_0^\infty ds s \times \left(\frac{1}{2} \int_{-1}^1 dv \int d^4 k \int d^4 p 2p.A(-k) \exp[-i(p + \frac{1}{2}k)^2 \frac{1}{2}(1-v)s] \right. \\ &\times 2p.A(k) \exp[-i(p - \frac{1}{2}k)^2 \frac{1}{2}(1+v)s] + \frac{1}{2} \int_{-1}^1 dv \int d^4 k \int d^4 p \frac{1}{4} \text{tr} \frac{1}{2} \sigma F \\ &\times \exp[-i(p + \frac{1}{2}k)^2 \frac{1}{2}(1-v)s] \frac{1}{2} \sigma F \exp[-i(p - \frac{1}{2}k)^2 \frac{1}{2}(1+v)s] \Big), \end{aligned} \quad (4.7)$$

then

$$\begin{aligned} J_1 &\simeq \frac{2ie^2}{(2\pi)^4} \int_0^\infty ds s \Big[(-2\pi^2 s^{-3}) \int d^4 k A_\mu(-k) A_\mu(k) \\ &- \int (i\pi^2 s^{-2}) d^4 k \frac{1}{2} F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv (1-v^2) e^{-\frac{ik^2(1-v^2)s}{4}} \Big]. \end{aligned} \quad (4.8)$$

Finally, we obtain J_1 in the first order approximation as

$$\begin{aligned} J_1 &\simeq \frac{e^2}{4\pi^2} \int_0^\infty s^{-2} ds \int d^4 k A_\mu(-k) A_\mu(k) \\ &+ \frac{e^2}{6\pi^2} \int_0^\infty s^{-1} ds \int d^4 k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ &- \frac{e^2}{8\pi^2} \int d^4 k F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 - v^2}. \end{aligned} \quad (4.9)$$

By replacing J , J_0 and J_1 in equation Eq.(3.8) we find

$$J_{kr} = J + J_0 + J_1$$

$$\begin{aligned} J_{kr} = & -\frac{e^2}{12\pi^2} \int_0^\infty ds s^{-1} \exp(-m^2 s) \int d^4 k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ & + \frac{e^2}{4\pi^2} \int d^4 k \frac{k^2}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{m^2 + \frac{k^2}{4}(1 - v^2)} \\ & + \frac{e^2}{12\pi^2} \int_0^\infty s^{-1} ds \int d^4 k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ & - \frac{e^2}{16\pi^2} \int d^4 k F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 - v^2}. \end{aligned} \quad (4.10)$$

This equation can be written in the following form:

$$J_{kr} = \frac{e^2}{16\pi^2} \int d^4 k F_{\mu\nu}(-k) F_{\mu\nu}(k) [I_1 + I_2 + I_3 + I_4], \quad (4.11)$$

where

$$\begin{aligned} I_1 &= -\frac{1}{3} \int_0^\infty ds s^{-1} \exp(-m^2 s), \quad I_3 = \frac{1}{3} \int_0^\infty s^{-1} ds, \\ I_2 &= \int_0^1 dv k^2 \frac{v^2(1 - \frac{1}{3}v^2)}{m^2 + \frac{k^2}{4}(1 - v^2)}, \quad I_4 = -\int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 - v^2}. \end{aligned}$$

The integrals I_1 , I_3 and I_4 are divergence. I_1 is:

$$I_1 = -\frac{1}{3} \int_0^\infty ds s^{-1} \exp(-m^2 s) = -\frac{1}{3} \Gamma(0).$$

By using the following relations:

$$\lim_{x \rightarrow 0} \Gamma(x) = \lim_{x \rightarrow 0} E_1(x),$$

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{nn!},$$

where γ is the Euler's constant, we can write

$$\Gamma(0) = -\gamma - \lim_{x \rightarrow 0} \ln x,$$

or

$$I_1 = \frac{1}{3} (\gamma + \lim_{x \rightarrow 0} \ln x).$$

The divergence form of I_3 is

$$I_3 = \frac{1}{3} \int_0^\infty s^{-1} ds = \frac{1}{3} \left(\lim_{\Lambda \rightarrow \infty} \ln \Lambda - \lim_{\mu \rightarrow 0} \ln \mu \right).$$

The I_4 divergency, which has been discussed in Appendix B, is as:

$$I_4 = - \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 - v^2} = \frac{1}{3} \lim_{\mu \rightarrow 0} \ln \mu - \frac{1}{3} \ln 2 + \frac{5}{9}. \quad (4.12)$$

Therefore, J_{kr} becomes:

$$\begin{aligned} J_{kr} = & \frac{e^2}{16\pi^2} \int d^4k F_{\mu\nu}(-k) F_{\mu\nu}(k) \times \left[\frac{1}{3}(\gamma + \lim_{x \rightarrow 0} \ln x) + \frac{1}{3} \left(\lim_{\Lambda \rightarrow \infty} \ln \Lambda - \lim_{\mu \rightarrow 0} \ln \mu \right) \right. \\ & \left. - \frac{1}{3} \left(- \lim_{\mu \rightarrow 0} \ln \mu + \ln 2 - \frac{5}{3} \right) + \int_0^1 dv k^2 \frac{v^2(1 - \frac{1}{3}v^2)}{m^2 + \frac{k^2}{4}(1 - v^2)} \right]. \end{aligned} \quad (4.13)$$

It's well known that

$$\lim_{\Lambda \rightarrow \infty} \ln \Lambda \equiv - \lim_{x \rightarrow 0} \ln x$$

then, we see that $W_{kr} = W^0 + J_{kr}$ reduces to

$$\begin{aligned} W_{kr} = & - \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ & \times \left[1 - \frac{\alpha}{4\pi} \left(\frac{k^2}{m^2} \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 + \frac{k^2}{4m^2}(1 - v^2)} - \frac{4}{3} \left(\ln 2 - \frac{5}{3} - \gamma \right) \right) \right], \end{aligned} \quad (4.14)$$

where we have added the action integral of the Maxwell field, which is expressed in momentum space by

$$W^{(0)} = - \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k),$$

and we took $\alpha = \frac{e^2}{4\pi}$. If we put $\alpha = \frac{1}{137}$ and $\gamma = 0.5772156649\dots$, finally, the total effective action is

$$\begin{aligned} W_{kr} = & - \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \\ & \times \left[0.9987989919 - \frac{\alpha}{4\pi} \frac{k^2}{m^2} \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 + \frac{k^2}{4m^2}(1 - v^2)} \right]. \end{aligned} \quad (4.15)$$

Now, one compares this result with standard solution Eq.(A.7).

This new kind of regularization may be utilized in the calculation of the Lamb-Shift and Magnetic-Anomaly [16].

5 Conclusion

We recall that the negative frequency solutions of the field equation is needed for quantizing the minimally coupled scalar field in de Sitter space. Contrary to the Minkowski space, the elimination of de Sitter negative norms in the minimally coupled states breaks the de Sitter invariance. Then, for restoring the de Sitter invariance, one needs to take into account the negative norm states i.e. the Krein space quantization. It provides a natural tool for eliminating the singularity in the QFT.

Here, it is found that in this approximation the theory is free of any divergence and the effective action coincides with standard solution. So, for QED, we see that this quantization eliminates the singularity in the theory without changing the physical content of the theory in the one-loop approximation. This method can be used as an alternative way for solving the non-renormalizability of quantum gravity in the linear approximation.

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A Schwinger method

In this appendix, in order to make the paper self-contained, we will present all of Schwinger's method as a major reference for our calculation. We discuss the approximate evaluation of

$$W^{(1)} = i\frac{1}{2} \int_0^\infty ds s^{-1} \exp(-im^2 s) \text{Tr} U(s),$$

by an expansion in powers of eA_μ and $eF_{\mu\nu}$. So, we can write

$$H = H_0 + H_1,$$

where

$$H_0 = p^2, \quad H_1 = -e(p.A + A.p) - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} + e^2 A^2.$$

To obtain the expansion of $\text{Tr} U(s)$ in powers of H_1 , we observe that $U(s)$ obeys the differential equation

$$i\partial_s U(s) = (H_0 + H_1)U(s).$$

The related operator

$$V(s) = U_0^{-1}(s)U(s),$$

where

$$U_0 = \exp(-iH_0 s),$$

is determined by

$$i\partial_s V(s) = U_0^{-1}(s)H_1 U_0(s)V(s), \tag{A.1}$$

and

$$V(0) = 1.$$

From Eq.(A.1) one can obtain

$$V(s) = 1 - i \int_0^s ds' U_0^{-1}(s') H_1 U_0(s') V(s'),$$

and construct the solution by iteration:

$$V(s) = 1 - i \int_0^s ds' U_0^{-1}(s') H_1 U_0(s') + (-i)^2 \int_0^s ds' U_0^{-1}(s') H_1 U_0(s') \times \int_0^{s'} ds'' U_0^{-1}(s'') H_1 U_0(s'') + \dots \tag{A.2}$$

On introducing new variables of integration, u_1, u_2, \dots , according to

$$s' = su_1, s'' = s'u_2, \dots,$$

we obtain the expansion

$$\begin{aligned} U(s) &= \exp(-iHs) = U_0(s) - is \int_0^1 du_1 U_0((1-u_1)s) H_1 U_0(u_1 s) + \dots \\ &(-is)^n \int_0^1 u_1^{n-1} du_1 \dots \int_0^1 du_n U_0((1-u_1)s) H_1 U_0(u_1(1-u_1)s) \dots \\ &\times U_0(u_1 u_2 \dots u_{n-1}(1-u_n)s) H_1 U_0(u_1 u_2 \dots u_n s) + \dots \end{aligned} \quad (\text{A.3})$$

Instead of taking the trace of this expression directly, which would involve further simplifications, we remark that

$$\text{Tr} U(s) - \text{Tr} U_0(s) = -is \int_0^1 d\lambda \text{Tr} [H_1 \exp(-i(H_0 + \lambda H_1)s)],$$

by the expansion of $\exp(-i(H_0 + \lambda H_1)s)$, one can write

$$\begin{aligned} \text{Tr} U(s) &= \text{Tr} U_0(s) + (-is) \text{Tr} [H_1 U_0(s)] + \frac{1}{2} (-is)^2 \int_0^1 du_1 \text{Tr} [H_1 U_0((1-u_1)s) H_1 U_0(u_1 s) + \dots \\ &\frac{(-is)^{n+1}}{n+1} \int_0^1 u_1^{n-1} du_1 \dots \int_0^1 du_n \text{Tr} [H_1 U_0((1-u_1)s) H_1 U_0(u_1(1-u_1)s) \dots \\ &\times U_0(u_1 u_2 \dots u_{n-1}(1-u_n)s) H_1 U_0(u_1 u_2 \dots u_n s)] + \dots \end{aligned}$$

We shall retain only the first nonvanishing field dependent terms in this expansion:

$$\begin{aligned} W^{(1)} &= \frac{1}{2} i e^2 \int_0^\infty ds s^{-1} \exp(-im^2 s) \times \left\{ -is \text{Tr} [A^2 \exp(-ip^2 s)] + \right. \\ &\frac{1}{2} (-is)^2 \int_{-1}^1 \frac{1}{2} dv \text{Tr} \left[(pA + Ap) \exp\left(-ip^2 \frac{1}{2}(1-v)s\right) \times (pA + Ap) \exp\left(-ip^2 \frac{1}{2}(1+v)s\right) \right] \\ &\left. + \frac{1}{2} (-is)^2 \int_{-1}^1 \frac{1}{2} dv \text{Tr} \left[\frac{1}{2} \sigma F \exp\left(-ip^2 \frac{1}{2}(1-v)s\right) \times \frac{1}{2} \sigma F \exp\left(-ip^2 \frac{1}{2}(1+v)s\right) \right] \right\}. \end{aligned} \quad (\text{A.4})$$

For convenience, the variable u_1 has been replaced by $\frac{1}{2}(1+v)$. The evaluation of these traces is naturally performed in a momentum representation. The matrix elements of the coordinate dependent field quantities depend only on momentum differences,

$$\langle P + \frac{1}{2}k | A_\mu | P - \frac{1}{2}k \rangle = \frac{1}{(2\pi)^4} \int dx e^{-ikx} A_\mu(x) \equiv (2\pi)^{-2} A_\mu(k)$$

and

$$\langle P | A_\mu^2 | P \rangle = \frac{1}{(2\pi)^4} \int dx A_\mu^2(x) = (2\pi)^{-4} \int dk A_\mu(-k) A_\mu(k).$$

Therefore

$$W^{(1)} = \frac{2ie^2}{(2\pi)^4} \int_0^\infty ds s^{-1} \exp(-im^2 s) \times \left\{ -is \int d^4 k A_\mu(-k) A_\mu(k) \int d^4 p \exp(-ip^2 s) + \right.$$

$$\begin{aligned}
& \frac{1}{2}(-is)^2 \int_{-1}^1 \frac{1}{2} dv \int d^4 k \int d^4 p 2p_\mu A_\mu(-k) \\
& \times \exp \left[-i \left(p + \frac{1}{2}k \right)^2 \frac{1}{2}(1-v)s \right] 2p_\nu A_\nu(k) \exp \left[-i \left(p - \frac{1}{2}k \right)^2 \frac{1}{2}(1+v)s \right] \\
& + \frac{1}{2}(-is)^2 \int_{-1}^1 \frac{1}{2} dv \int d^4 k \int d^4 p \frac{1}{4} \text{tr} \frac{1}{2} \sigma F \\
& \times \exp \left(-i \left(p + \frac{1}{2}k \right)^2 \frac{1}{2}(1-v)s \right) \frac{1}{2} \sigma F \exp \left[-i \left(p - \frac{1}{2}k \right)^2 \frac{1}{2}(1+v)s \right] \Big\}. \quad (\text{A.5})
\end{aligned}$$

We thus encounter the elementary integrals

$$\begin{aligned}
& \int d^4 p \exp(-ip^2 s) = -i\pi^2 s^{-2}, \\
& \int d^4 p \exp \left[-i \left(p^2 + \frac{k^2}{4} \right) s + ipkvs \right] = -i\pi^2 s^{-2} \exp \left[-i \frac{k^2}{4} (1-v^2) s \right], \\
& \int d^4 p p_\mu p_\nu \exp \left[-i \left(p^2 + \frac{k^2}{4} \right) s + ipkvs \right] = -i\pi^2 s^{-2} \left(-\frac{i}{2} s^{-1} \delta_{\mu\nu} + \frac{1}{4} v^2 k_\mu k_\nu \right) \exp \left[-i \frac{k^2}{4} (1-v^2) s \right].
\end{aligned}$$

It is convenient to replace the $\delta_{\mu\nu}$ term of the last integral by an expression which is equivalent to it in virtue of the integration with respect to v . Now

$$\int_{-1}^1 \frac{1}{2} dv \exp \left[-i \frac{k^2}{4} (1-v^2) s \right] = 1 - is \frac{1}{2} k^2 \int_{-1}^1 \frac{1}{3} dv v^2 \exp \left[-i \frac{k^2}{4} (1-v^2) s \right],$$

so that, effectively

$$\begin{aligned}
& \int d^4 p p_\mu p_\nu \exp \left[-i \left(p^2 + \frac{k^2}{4} \right) s + ipkvs \right] = -\frac{1}{2} \pi^2 s^{-3} \delta_{\mu\nu} \\
& + \frac{i}{4} \pi^2 s^{-2} v^2 (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \exp \left[-i \frac{k^2}{4} (1-v^2) s \right].
\end{aligned}$$

On inserting the values of the various integrals, and noticing that

$$(\delta_{\mu\nu} k^2 - k_\mu k_\nu) A_\mu(-k) A_\nu(k) = \frac{1}{2} F_{\mu\nu}(-k) F_{\mu\nu}(k),$$

we obtain immediately the gauge invariant form (with $s \rightarrow -is$)

$$W^{(1)} = -\frac{e^2}{4\pi^2} \int d^4 k \frac{1}{2} F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv (1-v^2) \int_0^\infty ds s^{-1} \exp \left[- \left(m^2 + \frac{k^2}{4} (1-v^2) \right) s \right].$$

This has been achieved without any special device, other than that of reserving the proper time integration to the last. A significant separation of terms is produced by a partial integration with respect to v , according to

$$\int_0^1 dv (1-v^2) \int_0^\infty ds s^{-1} \exp[-(m^2 + \frac{k^2}{4} (1-v^2))s] = \frac{2}{3} \int_0^\infty ds s^{-1} \exp(-m^2 s) -$$

$$\frac{1}{2}k^2 \int_0^1 dv (v^2 - \frac{1}{3}v^4) \int_0^\infty ds \exp[-(m^2 + \frac{k^2}{4}(1-v^2))s].$$

Adding the action integral of the Maxwell field, which is expressed in momentum space by

$$W^{(0)} = - \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k),$$

we obtain the modified integral,

$$\begin{aligned} W = & - \left[1 + \frac{e^2}{12\pi^2} \int_0^\infty ds s^{-1} \exp(-m^2 s) \right] \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) + \\ & \frac{e^2}{4\pi^2} \int d^4k \frac{k^2}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{m^2 + \frac{k^2}{4}(1-v^2)}. \end{aligned} \quad (\text{A.6})$$

The field strength and charge renormalization produces the finite gauge invariant result.

$$W = - \int d^4k \frac{1}{4} F_{\mu\nu}(-k) F_{\mu\nu}(k) \left[1 - \frac{\alpha}{4\pi} \frac{k^2}{m^2} \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 + \frac{k^2}{4m^2}(1-v^2)} \right]. \quad (\text{A.7})$$

B Calculations

We briefly present some calculations and simplifications which has been used in this paper.

In this part we'll bring some calculations to simplify the logarithmic functions which was used in section(4). It is easy to see that

$$K_+ = \ln \left(1 + \frac{V}{k^2(k^2 - m^2) - V} \right) = - \ln \left[1 - \frac{V}{k^2(k^2 - m^2)} \right].$$

Then, we can write

$$\begin{aligned} \ln \left(1 - \frac{V}{k^2(k^2 - m^2)} \right) &= \ln \left[1 + \frac{V}{m^2} \left(\frac{1}{k^2} - \frac{1}{k^2 - m^2} \right) \right] \\ &= \ln \left[\left(1 + \frac{V}{m^2 k^2} \right) \left(1 - \frac{V}{m^2(k^2 - m^2)} \right) + \frac{V^2}{m^4 k^2(k^2 - m^2)} \right], \end{aligned} \quad (\text{B.8})$$

and finally we get to

$$K_+ = - \ln \left(1 + \frac{V}{m^2 k^2} \right) - \ln \left(1 - \frac{V}{m^2(k^2 - m^2)} \right) - \ln \left(1 + \frac{V^2}{(m^2 k^2 + V)(m^2 k^2 - m^4 - V)} \right). \quad (\text{B.9})$$

Now, we calculate

$$\begin{aligned} K_- &= \ln \left(1 - \frac{V}{k^2(k^2 - m^2) - V} \right) = \ln \left(\frac{k^2(k^2 - m^2) - 2V}{k^2(k^2 - m^2) - V} \right) \\ &= \ln \left(1 - \frac{2V}{k^2(k^2 - m^2)} \right) - \ln \left(1 - \frac{V}{k^2(k^2 - m^2)} \right). \end{aligned} \quad (\text{B.10})$$

So, we obtain

$$K_- = \ln \left(1 + \frac{2V}{m^2 k^2} \right) + \ln \left(1 - \frac{2V}{m^2 (k^2 - m^2)} \right) + \ln \left(1 + \frac{4V^2}{(m^2 k^2 + 2V)(m^2 k^2 - m^4 - 2V)} \right) - \ln \left(1 - \frac{V}{k^2 (k^2 - m^2)} \right). \quad (\text{B.11})$$

Now, we would like to present the calculation of I_4 in (4.4):

$$I_4 = - \int_0^1 dv \frac{v^2(1 - \frac{1}{3}v^2)}{1 - v^2} = -\frac{1}{2} \int_0^1 dv \left[\frac{v^2(1 - \frac{1}{3}v^2)}{1 - v} + \frac{v^2(1 - \frac{1}{3}v^2)}{1 + v} \right]. \quad (\text{B.12})$$

By using the following relations:

$$\frac{1}{1 - v} = \sum_{n=0}^{\infty} v^n, \quad \frac{1}{1 + v} = \sum_{n=0}^{\infty} (-v)^n,$$

we obtain

$$\begin{aligned} I_4 &= -\frac{1}{2} \int_0^1 dv \sum_{n=0}^{\infty} \left[v^{n+2} - \frac{1}{3} v^{n+4} + (-v)^{n+2} - \frac{1}{3} (-v)^{n+4} \right] \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \left[\int_0^1 dv v^{n+2} - \frac{1}{3} \int_0^1 dv v^{n+4} + \int_0^1 dv (-v)^{n+2} - \frac{1}{3} \int_0^1 dv (-v)^{n+4} \right] \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{v^{n+3}}{n+3} - \frac{1}{3} \frac{v^{n+5}}{n+5} + \frac{(-1)^n v^{n+3}}{n+3} - \frac{1}{3} \frac{(-1)^n v^{n+5}}{n+5} \right]_0^1. \end{aligned} \quad (\text{B.13})$$

And so, we can rewrite as

$$\begin{aligned} I_4 &= -\frac{1}{2} \left\{ \left[-\ln(1 - v) - v - \frac{v^2}{2} \right] + \frac{1}{3} \left[\ln(1 - v) + v + \frac{v^2}{2} + \frac{v^3}{3} + \frac{v^4}{4} \right] \right. \\ &\quad \left. + \left[\ln(1 + v) - v + \frac{v^2}{2} \right] - \frac{1}{3} \left[\ln(1 + v) - v + \frac{v^2}{2} - \frac{v^3}{3} + \frac{v^4}{4} \right] \right\}_0^1. \end{aligned}$$

Finally, we have

$$\begin{aligned} I_4 &= - \left[-\frac{1}{3} \ln(1 - v) + \frac{1}{3} \ln(1 + v) - \frac{2}{3} v + \frac{1}{9} v^3 \right]_0^1 \\ &= \frac{1}{3} \lim_{\mu \rightarrow 0} \ln \mu - \frac{1}{3} \ln 2 + \frac{5}{9}. \end{aligned} \quad (\text{B.14})$$

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